Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Applied Mathematics and Mechanics 70 (2006) 65–72
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Three-dimensional non-linear oscillations of a rod with hinged supports☆

A.I. Munitsyn

Ivanova

Received 18 December 2003

Abstract

The free and forced flexural oscillations of a rod with hinged supports are investigated analytically and numerically. The geometrical non-linearity due to the change in the length of the central line of the rod accompanying its three-dimensional motion is taken into account. The oscillations of a rod with different natural frequencies in two mutually perpendicular directions as a consequence of the variance in the flexural stiffnesses of the rod or the stiffnesses of the supports in the different directions, are considered. It is shown in the case of natural oscillations that, together with two planar forms of motion, a form exists when a certain threshold value is exceeded, which corresponds to the motion of the cross-sections of the rod in a circle. The amplitude-frequency and phase-frequency characteristics of the system are constructed and qualitatively investigated in the neighbourhood of the principal resonance.

A relation between the oscillations in different directions, which leads to the existence of both planar forms of motion as well as three-dimensional forms, in which points of the string move in a circle, has been discovered in investigations of the oscillations of a string with fixed supports.[1–3] In the case of forced oscillations, a range of frequencies exists in the neighbourhood of the principal resonance at which stable parametric oscillations occur in the plane orthogonal to the action of the generating force and the overall motion of the points occurs along an ellipse.

The existence of planar and three-dimensional forms of motion has also been found in the numerical investigation of the three-dimensional oscillations of an inelastic[4] and an elastic[5] thread with a tensioning device, with the difference that the system has a weak elastic rather than a stiff elastic characteristic.

The equations which describe the planar flexural oscillations of a rod and a string with fixed supports reduce to the same form in dimensionless variables[6] and, in this case, the qualitative results will be the same. The non-linear oscillations of shallow curvilinear rods in the plane of curvature has been considered earlier in Ref. 7.

1. Formulation of the problem

Suppose the central axis of a rod in the undeformed state coincides with the x axis of a rectangular system of coordinates and the principal axes of inertia of the cross-section are parallel to the y and z axes. The coordinates $x=0$ and $x=L$ correspond to the ends of the rod. We will denote the displacements of the points of the centre line of the rod by u, v and w. In the technical theory of the flexure of a rod, the longitudinal deformation at a point of the cross-section

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 1, pp. 72–80, 2005.
E-mail address: munitsyn@rambler.ru.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2006.03.009

with coordinates y and z is equal to

$$\varepsilon_x = \varepsilon_0 - y\upsilon'' - zw'', \quad \varepsilon_0 = u' + \frac{1}{2}(\upsilon'^2 + w'^2) \tag{1.1}$$

A derivative with respect to x is denoted by a prime, ε_0 is the deformation of the centre line of the rod and we neglect the term $u'^2/2$. This relationship enables us to take account of the elongation of the centre line due to displacements along the y and z axes. The expressions for the potential and kinetic energy of the rod have the form

$$\begin{aligned} U &= \frac{EF}{2}\int_0^L \varepsilon_0^2 dx + \frac{EJ_z}{2}\int_0^L \upsilon''^2 dx + \frac{EJ_y}{2}\int_0^L w''^2 dx \\ T &= \frac{\rho F}{2}\int_0^L (\dot{u}^2 + \dot{\upsilon}^2 + \dot{w}^2) dx \end{aligned} \tag{1.2}$$

where E and ρ are the modulus of elasticity and the density of the rod material, F is the cross-section area, J_y and J_z are the axial moments of inertia of the cross-section and a time derivative is denoted by a dot. Torsion is not taken into account since, henceforth, the flexural oscillations of a thin rod in the neighbourhood of the first natural frequency are principally considered.

We us the Hamilton–Ostrogradskii principle and obtain a system of three partial differential equations

$$\rho F\ddot{u} - EF\varepsilon_0' = 0, \ \rho F\ddot{\upsilon} + EJ_z\upsilon^{IV} - EF(\varepsilon_0\upsilon')' = 0, \ \rho F\ddot{w} + EJ_y w^{IV} - EF(\varepsilon_0 w')' = 0 \tag{1.3}$$

Boundary conditions must be added to system (1.3) and, for the special case of a rod with hinged supports at the ends, we have the boundary conditions

$$u = \upsilon = w = \upsilon'' = w'' = 0 \quad \text{when} \quad x = 0, L \tag{1.4}$$

2. Natural oscillations

In the case of a thin rod, the normal modes of the longitudinal and flexural oscillations are separated: $\omega_x \gg \omega_y$, $\omega_x \gg \omega_z$. Consequently, the quantity $\ddot{u}$ can be neglected in the first of Eq. (1.3) when investigating the oscillations in the neighbourhood of the first frequency, from which it follows that ε_0 is independent of the x coordinate. Assuming that there is no relative longitudinal displacement of the supports, according to boundary conditions (1.4), we obtain

$$\varepsilon_0 = \frac{1}{2}\int_0^1 (\upsilon'^2 + w'^2) dx \tag{2.1}$$

Substituting this expression into the remaining two equations of (1.3), we have two integrodifferential equations in the dimensionless variables

$$\pi^4\ddot{\upsilon} + \upsilon^{IV} - 4\gamma\varepsilon_0\upsilon'' = 0, \quad \pi^4\ddot{w} + cw^{IV} - 4\gamma\varepsilon_0 w'' = 0 \tag{2.2}$$

Here,

$$\gamma = FL^2/(4J_z), \quad c = J_y/J_z$$

All of the displacements and the x coordinate are divided by the length of the rod L and differentiation with respect to the dimensionless time, which is obtained from an initial multiplication by the frequency of the small natural oscillations in the xy plane

$$\omega_1 = (\pi/L)^2[EJ_z/(\rho F)]^{1/2}$$

is denoted by dots.

We shall confine ourselves to the case of the single-mode approximation and represent the solution in the form

$$v(x,t) = \varphi_1(t)\sin\pi x, \quad w(x,t) = \varphi_2(t)\sin\pi x$$

Substitution into (2.2) leads to a system with two degrees of freedom

$$\ddot{\varphi}_k + [1+(k-1)\varepsilon\delta]\varphi_k + \varepsilon\gamma(\varphi_1^2+\varphi_2^2)\varphi_k = 0, \quad k = 1,2 \tag{2.3}$$

where a small parameter ε has been introduced and the substitution $c = 1 + \varepsilon\delta$ has been made, that is, the non-linearity of the system and the difference between the natural frequencies are assumed to be asymptotically small, which enables us to use efficient methods of non-linear mechanics.[8,9]

System (2.3) has a strong cubic non-linearity and, in view of the smallness of $\varepsilon\delta$, phenomena of the internal resonance type are characteristic of this system. A detailed investigation of a similar system has been carried out in Ref. 10 using the multiscale method.

Making the change of variables

$$\varphi_k = a_k\cos(t+\alpha_k), \quad k = 1,2 \tag{2.4}$$

in system (2.3) and using the method of averaging, we obtain a fairly simple system of equations in the slow variables

$$\begin{gathered}\dot{a}_k = \frac{1}{8}\varepsilon\gamma a_1^k a_2^{3-k}S_1, \quad \dot{\alpha}_k = \frac{1}{2}\varepsilon\delta(k-1) + \frac{3}{8}\varepsilon\gamma a_k^2 + \frac{1}{8}\varepsilon\gamma a_{3-k}^2(2+S_2)\\ S_1 = \sin 2(\alpha_1-\alpha_2), \quad S_2 = \cos 2(\alpha_1-\alpha_2), \quad k = 1,2\end{gathered} \tag{2.5}$$

where a_k and α_k are the amplitudes and phases of the partial oscillations, and the free oscillations are considered in a small neighbourhood of a single frequency. System (2.5) has three solutions.

(1) $a_2 = 0$. In this case, system (2.5) takes the form

$$\dot{a}_1 = 0, \quad \dot{\alpha}_1 = \frac{3}{8}\varepsilon\gamma a_1^2$$

and its solution is

$$a_1 = \text{const}, \quad \alpha_1 = \frac{3}{8}\varepsilon\gamma a_1^2 t + \alpha_{10}$$

Substituting the values of a_1 and α_1 obtained into relations (2.4), we obtain the oscillations of the rod in the xy plane

$$\varphi_1 = a_1\cos((1+\varepsilon\lambda)t+\alpha_{10}), \quad \varphi_2 \equiv 0 \tag{2.6}$$

where we have introduced the notation

$$\lambda = \frac{3}{8}\gamma a_1^2 \tag{2.7}$$

Relation (2.7) describes the well-known dependence of the amplitude on the frequency of the natural oscillations of a rod in a plane,[6,9] which is represented by curve *1* in Fig. 1 for $\gamma = 1$.

(2) $a_1 \equiv 0$. The solution of system (2.5) is

$$a_2 = \text{const}, \quad \alpha_2 = \frac{1}{2}\varepsilon\delta + \frac{3}{8}\varepsilon\gamma a_2^2 t + \alpha_{20}$$

Substituting a_2 and α_2 into relations (2.4), we obtain the oscillations of the rod in the xz plane

$$\varphi_2 = a_2\cos((1+\varepsilon\lambda)t+\alpha_{20}), \quad \varphi_1 \equiv 0 \tag{2.8}$$

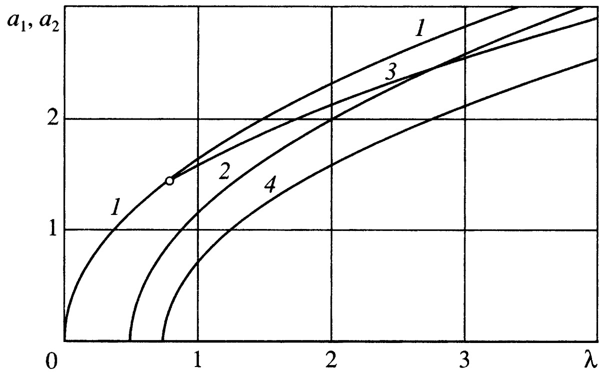

Fig. 1.

where

$$\lambda = \frac{1}{2}\delta + \frac{3}{8}\gamma a_2^2 \tag{2.9}$$

Curve *2* ($\gamma = 1$, $\delta = 1$) in Fig. 1 corresponds to relation (2.9).

The solutions (2.6) and (2.8) correspond to the planar oscillations of a rod in two mutually orthogonal planes. Since the flexural stiffnesses are different ($J_y \neq J_z$), the natural frequencies differ by an amount $\delta/2$.

(3) $\alpha_1 - \alpha_2 = \pm\pi/2$. In this case, $S_1 = 0$ and $S_2 = -1$ in system (2.5) and its solution has the form

$$a_k = \text{const}, \quad \alpha_k = (1 + \varepsilon\lambda)t + \alpha_{k0}, \quad k = 1, 2$$

The parameter λ is determined from the system of equations

$$\lambda = \frac{1}{2}\delta(k-1) + \frac{3}{8}\gamma a_k^2 + \frac{1}{8}\gamma a_{3-k}^2, \quad k = 1, 2$$

the solution of which gives the amplitude-frequency dependence of the three-dimensional oscillations of the rod, which corresponds to motion of the points of the centre line along an ellipse in the *yz* plane

$$a_1(\lambda) = (2(\lambda + \delta/4)/\gamma)^{1/2}, \quad a_2(\lambda) = (2(\lambda - 3\delta/4)/\gamma)^{1/2} \tag{2.10}$$

The domain of frequencies exceeding $\lambda = 3\delta/4$ corresponds to real values of a_2 and, for three-dimensional oscillations to occur, it is necessary that the amplitude a_1 should exceed the critical value $a_* = (2\delta/\gamma)^{1/2}$.

Curves *3* ($a_1(\lambda)$) and *4* ($a_2(\lambda)$) in Fig. 1 correspond to relations (2.10). Only planar oscillations are possible in the system being considered when $a_1 < a_*$ and a three-dimensional form of motion of the rod exists, together with the planar motions, when $a_1 < a_*$. Hence, depending on the initial conditions, the natural oscillations of the rod are either oscillations in mutually orthogonal planes with two different frequencies, which are independent of one another, or a further three-dimensional form of motion with a third frequency of the oscillations is added to them. In the general case, there are three harmonics with close frequencies in the resulting motion. In the case when $\delta = 0$, the qualitative pattern of the oscillations corresponds to the solution obtained for the vibrations of a string[1,4] which can be represented as a sum of planar and three-dimensional motions.

3. Forced oscillations

Suppose external harmonic loads act on the rod in mutually perpendicular directions with a phase difference $\psi = \psi_2 - \psi_1$. The normalized external actions, which correspond to a first mode $\varepsilon f_1 \cos[(1 + \varepsilon\lambda)t + \psi_1]$ in the *xy* plane and $\varepsilon f_2\cos[(1+\varepsilon\lambda)t+\psi_2]$ in the xz plane, are assumed to be small, and we can confine ourselves to considering oscillations in a small neighbourhood of the principal resonance. The equations of motion have the form

$$\ddot{\varphi}_k + \varepsilon\eta\dot{\varphi}_k + [1+(k-1)\varepsilon\delta]\varphi_k + \varepsilon\gamma(\varphi_1^2+\varphi_2^2)\varphi_k = \\ = \varepsilon f_k\cos((1+\varepsilon\lambda)t+\psi_k), \quad k = 1, 2 \tag{3.1}$$

where η is a dimensionless dissipation coefficient.

Making the change of variables

$$\varphi_k = a_k\cos[(1+\varepsilon\lambda)t+\alpha_k], \quad k = 1, 2$$

in system (3.1), in the first approximation of the method of averaging we obtain the equations

$$\begin{aligned} \dot{a}_k &= -\frac{1}{2}\varepsilon f_k\sin(\alpha_k+\psi_k) + \frac{1}{8}\varepsilon\gamma a_1^k a_2^{3-k}S_1 - \frac{1}{2}\varepsilon\eta a_k \\ \dot{\alpha}_k &= \frac{1}{2}\varepsilon[\delta(k-1)-2\lambda] - \frac{1}{2a_k}\varepsilon f_k\cos(\alpha_k-\psi_k) + \\ &+ \frac{3}{8}\varepsilon\gamma a_k^2 + \frac{1}{8}\varepsilon\gamma a_{3-k}^2(2+S_2); \quad k = 1, 2 \end{aligned} \tag{3.2}$$

We shall next consider steady forced oscillations which correspond to zero left-hand sides of Eqs. (3.2). An analytical solution of system (3.2) can only be obtained in individual specific cases. We shall use the method of the continuation of a solution with respect to a small parameter for the numerical solution of the corresponding system of non-linear algebraic equations. If an approximate solution of the system of equations $r = (a_1^k, a_2^k, \alpha_1^k, \alpha_2^k)^T$ is known for a certain value λ^k, then, for a value $\lambda^{k+1} = \lambda^k + \Delta\lambda^k$, the approximate solution can be represented in the form $r^{k+1} = r^k + \Delta r^k$. Substituting into system (3.2) and linearizing the resulting equations, we determine the increments in the unknowns from the system

$$G^k\Delta r^k = p^k + R^k \tag{3.3}$$

where R^k is the vector of the discrepancy in the preceding step of the solution, $p^k = (0, 0, \lambda^k, \lambda^k)^T$ and the elements of the matrix G have the form (the superscript k is henceforth omitted)

$$\begin{aligned} g_{ii} &= \frac{(-1)^{i+1}}{8}\gamma a_{3-i}^2 S_1 - \frac{1}{2}\eta \\ g_{i,i+2} &= \frac{1}{4}\gamma a_1^i a_2^{3-i}S_2 - \frac{1}{2}f_i\cos(\alpha_i+\psi_i) \\ g_{i+2,i} &= \frac{1}{2a_i^2}f_i\cos(\alpha_i+\psi_i) + \frac{3}{4}\gamma a_i \\ g_{i+2,i+2} &= \frac{(-1)^i}{4}\gamma a_{3-i}^2 S_1 + \frac{1}{2a_i}f_i\sin(\alpha_i+\psi_i) \\ g_{i,3-i} &= \frac{(-1)^{i+1}}{4}\gamma a_1 a_2 S_1, \quad g_{i,5-i} = -\frac{1}{4}\gamma a_i a_{3-i}^2 S_2 \\ g_{i+2,3-i} &= \frac{1}{4}\gamma a_{3-i}(2+S_2), \quad g_{i+2,5-i} = \frac{(-1)^{i+1}}{4}\gamma a_{3-i}^2 S_1; \quad i = 1, 2 \end{aligned} \tag{3.4}$$

Hence, the solution of the system of non-linear equations (3.2) reduces to the solution of the sequence of systems of linear equations (3.3). At each step in the calculations, the magnitude of the discrepancy is checked and, if the relative error exceeds the specified accuracy, then the step in the variable being varied is reduced. At the branching points of the solutions, the variable which has the increment with the greatest modulus in the preceding step is adopted

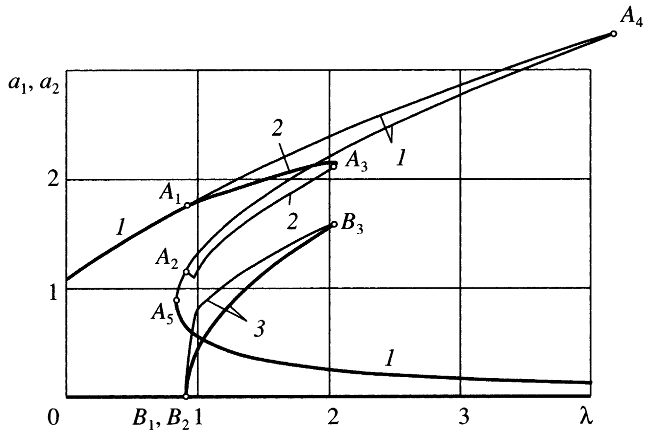

Fig. 2.

as the independent parameter, which enables us to find all of the existing solutions and to construct the multivalued amplitude-frequency and phase-frequency characteristics.

To investigate the stability of the solutions obtained using the second Lyapunov method, we will consider a certain perturbed solution of the system of equations (3.2) $\hat{r} = (t) = r(t) + \Delta r(t)$. After substitution into system (3.2) and linearization, we obtain the equations of the perturbed motion in the first approximation $\Delta r = G\Delta r$, where the matrix G is identical to the matrix appearing in Eq. (3.3). According to the theorems of stability using a first approximation, the sign of the real part of all of the eigenvalues of the matrix G enables us to draw a conclusion regarding the stability of the solution. Since the matrix is asymmetric, the QR-algorithm[11] was used to solve the eigenvalue problem.

4. Results of numerical modelling

All the calculations were carried out for $\gamma = 1$ and $\delta = 1$. The relations $a_1(\lambda)$ and $a_2(\lambda)$ are represented in Fig. 2 for $f_1 = 1, f_2 = 10^{-6}$, $\eta = 0.3$, $\psi = 0$, which corresponds to the case of the excitation of planar oscillations corresponding to a smaller flexural stiffness. The value of f_2 was taken to be non-zero in order to avoid an overflow error in Eqs. (3.2) and (3.4) on division by a_2. The segments corresponding to stable solutions are shown by the solid lines.

The known relation $a_1(\lambda)$, which corresponds to planar oscillations ($a_2(\lambda) \equiv 0$), is represented by curve *1*. Its construction was started with a fairly small detuning of the frequencies and zero initial approximations for the amplitude a_2 and the phase α_2. There are two branching points on curve *1*: A_1 and A_2 to which the points B_1 and B_2 of the relation $a_2(\lambda)$ correspond (the points B_1 and B_2 merge in Fig. 2).

In the case when one of the two points, (A_1, B_1), for example, was adopted as the initial approximation, a solution $a_1(\lambda)$ (curve *2*) and $a_2(\lambda)$ (curve *3*) was constructed, which corresponds to a three-dimensional form of the motion of the centre line of rod. When the frequency of the excitation is increased, the planar oscillations smoothly become three-dimensional oscillations for which the points of the axial line of the rod describe ellipses in a plane which is orthogonal to the x axis, and the phase difference $\alpha_1 - \alpha_2$ is equal to $\pi/2$. A further increase in the frequency leads to a jump at the point (A_3, B_3) to the planar form of motion of the rod. When the excitation frequency is reduced, the three-dimensional form of motion is not obtained. A solution, which corresponds to the planar form (curve *1*), can be obtained from the solution of the planar problem, where the segment between the point A_1 and the maximum of the amplitude-frequency characteristic at the point A_4 will correspond to a stable solution.

Analogous results for the case of the excitation of oscillation in a plane, corresponding to a greater flexural stiffness when $f_1 = 10^{-6}, f_2 = 1$, $\psi = 0$ are shown in Fig. 3 ($\eta = 0.18$) and Fig. 4 ($\eta = 0.22$). The relation $a_2(\lambda)$ of the planar form of motion corresponds to curve *1*, and the relations $a_2(\lambda)$ and $a_1(\lambda)$ of the three-dimensional form correspond to curves *2* and *3*. Unlike the preceding case, investigation of the stability of the solution of the planar problem leads to the correct result everywhere apart from a small segment in the neighbourhood of the points (A_1, B_1) and (A_2, B_2). When the frequency is increased, the planar form of motion when $\eta = 0.18$ passes smoothly into the three-dimensional form and, at the point (A_3, B_3), there is a jump into one of the stable branches of the planar form 1. In experiments, the jump

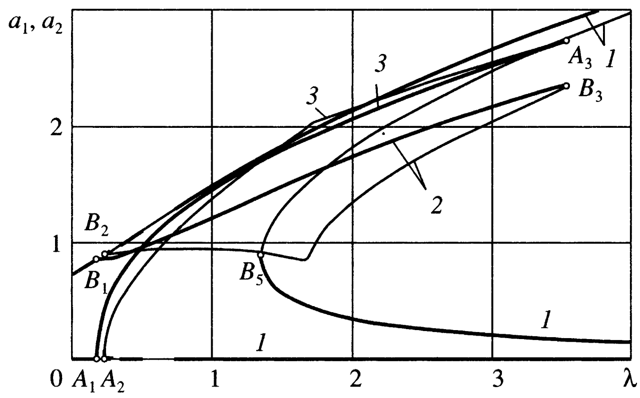

Fig. 3.

is usually observed into the upper branch of curve *1* and the subsequent oscillations occur in the plane of action of the load. The point B_4, which corresponds to the maximum amplitude of the planar form of motion, is not shown in Figs. 3 and 4. The existence of from one to three stable solutions, two planar motions and a single three-dimensional motion, is possible over different frequency ranges.

When the dissipation parameter is increased, the points (A_1, B_1) and (A_2, B_2) converge and, when $\eta > 0.191$, curves *2* and *3* become closed as in Fig. 4. In this case, at the point (A_6, B_6) there is a stable excitation of the parametric oscillations in the *xy* plane. There are no stable steady-state solutions of system (3.2) in the narrow range of frequency detuning between the points (A_7, B_7) and (A_6, B_6).

A further increase in dissipation leads to the fact that the loops formed by curves *2* and *3* become smaller and disappear when $\eta > 0.252$. Three-dimensional forms of motion do not exist for larger values of η.

The amplitude-frequency characteristics $a_1(\lambda)$, $a_2(\lambda)$ for $f_1 = f_2 = 0.707$, $\eta = 0.25$, $\psi = 0$, which correspond to the case when the direction of the loading vector makes an angle of $\pi/4$ with the principal axes of inertia of the section, are shown in Fig. 5. It follows from an analysis of the phase-frequency characteristics under pre-resonance conditions that planar oscillations occur during which $a_1 > a_2$ and then, at the points of inflection of the curves, there is a smooth transition into the three-dimensional form of oscillations up to the point (A_3, B_3). In the trans-resonance state to the right of the point (A_5, B_5), planar oscillations occur and $a_1 < a_2$. For the given case of loading, it is characteristic for there to be a stable segment bounded by the points (A_2, B_2) and (A_4, B_4) in which a transition is observed from a three-

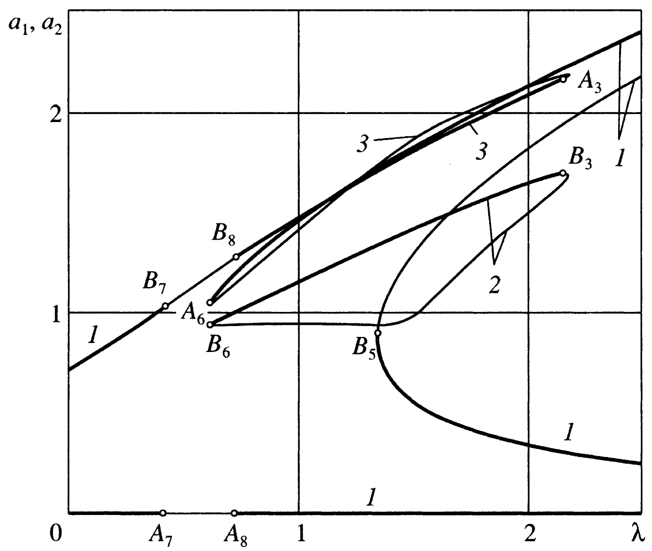

Fig. 4.

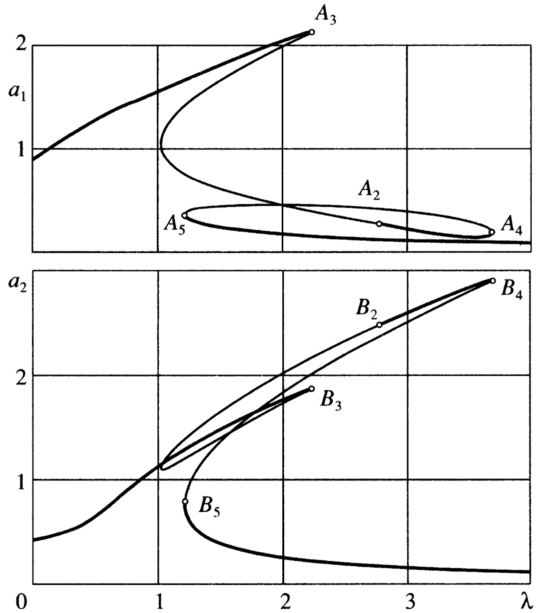

Fig. 5.

dimensional form of motion to a planar form of motion, and oscillations occur in practice in the plane corresponding to greater flexural stiffness of the rod. This segment is isolated from the remaining stable solutions and is only realized when there are external perturbations.

In all of the cases considered, two relations $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ correspond to each solution $a_1(\lambda)$, $a_2(\lambda)$. In the case of the three-dimensional form of motion, the second solution corresponds to a rotation of the section of the rod in the opposite direction and, in the case of the planar form of motion, it is unstable.

References

1. Akulenko LD, Nesterov SV. The non-linear vibrations of a string. *Izv Ross Akad Nauk MTT* 1993;**4**:87–92.
2. Akulenko LD, Nesterov SV. Forced non-linear vibrations of a string. *Izv Ross Akad Nauk MTT* 1996;**1**:17–24.
3. Akulenko LD, Kostin GV, Nesterov SV. The effect of dissipation on the three-dimensional non-linear vibrations of a string. *Izv Ross Akad Nauk MTT* 1997;**1**:19–28.
4. Munitsyn AI. Non-linear vibrations of a thread with a tensioning device. *Izv Ross Akad Nauk MTT* 2001;**2**:24–30.
5. Munitsyn AI. The three-dimensional non-linear vibrations of an elastic thread with a tensioning device. *Problemy Mashinostroyeniya i Nadezhnosti Mashin* 2001;**2**:21–8.
6. Kauderer H. *Nichtlineare mechanik*. Berlin: Springer; 1958.
7. Grigolyuk EI. Non-linear oscillations and the stability of shallow plates and shells. *Izv Akad Nauk SSSR Otd Tekhn Nauk* 1955;**3**:33–68.
8. Bogolyubov NN, Mitropol'skii YuA. *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Moscow: Nauka; 1974.
9. Nayfeh AH. *Introduction to Perturbation Techniques*. New York: Wiley; 1981.
10. Ladygina YeV, Manevich AI. The free oscillations of a non-linear cubic system with two degrees of freedom and close natural frequencies. *Prikl Mat Mekh* 1993;**57**(2):40–9.
11. Ikramov KHD. *The Asymmetric Eigenvalue Problem*. Moscow: Nauka; 1991.

Translated by E.L.S.